Journal of Al-Qadisiyah for computer science and mathematics
Vol.5 No.2 year 2013

Page8-21

On generalization of a subclass of p-valent harmonic functions

[1]*Waggas Galib Atshan* & [2]*Asraa Abdul Jaleel Husien*

Department of Mathematics College of Computer Science and Mathematics
University of Al-Qadisiya , Diwaniya – Iraq
E-mail: [1]waggashnd@gmail.com, waggas_hnd@yahoo.com
[2] asraalsade1@gmail.com, asra.alsade@yahoo.com

Recived : 9 /5 /2013 Revised: 15 /7 / 2013 Accepted: 23 /7 / 2013

Abstract. In the present paper, we introduce a generalization of subclass of p-valent harmonic functions in the open unit disk U, we obtain various properties for this subclass, like, coefficient bounds, extreme points, distortion theorem, convex set, convolution property and convex combinations.

Keywords and phrases: multivalent function, harmonic function, extreme points, distortion bounds, convex set, convolution, convex combinations.

Introduction: A continuous complex-valued function $f = u + iv$ defined in a simple connected complex domain D is said to be harmonic in D, if both u and v are real harmonic in D. Let $f = h + \bar{g}$ be defined in any simply connected domain, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D. (See Clunie and Sheil-Small [6]).

Denote by *H(p)* the class of functions $f = h + \bar{g}$ that are harmonic p- valent and sense-preserving in the unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$. For $f = h + \bar{g} \in H(p)$, we may express the analytic functions h and g as

$$h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n , \qquad g(z) = \sum_{n=p}^{\infty} b_n z^n , \quad |b_p| < 1. \qquad (1.1)$$

Let *A(p)* denote the subclass of *H(p)* consisting of functions $f = h + \bar{g}$, where h and g are given by

$$h(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n , g(z) = -\sum_{n=p}^{\infty} b_n z^n , (a_n \geq 0, b_n \geq 0, |b_p| < 1). \qquad (1.2)$$

We introduce here a class $H_q^{\beta}(p,\alpha)$ of harmonic functions of the form (1.1) that satisfy the inequality

$$Re\left\{(1-\beta\ )\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}+\beta\frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-(q+1)}}\right\} \geq \frac{\alpha}{\delta(p,q+1)}, \qquad (1.3)$$

where

$$f^{(q)}(z)=\delta(p,q)z^{p-q}+\sum_{n=p+1}^{\infty}\delta(n,q)\,a_n\,z^{n-q},$$

$$\delta(i,j)=\frac{i!}{(i-j)!}=\begin{cases}1 & j=0\\ i(i-1)\ldots(i-j+1) & j\neq 0,\end{cases}$$

$0\leq\alpha<p, p\in N, q\in N_0=N\cup\{0\}, p>q$ and $\beta\geq 0$.

We deem it worthwhile to point here the relevance of the function class $H_q^{\beta}(p,\alpha)$ with those classes of functions which have been studied recently. Indeed, we observe that:

(i) $H_0^1(1,0)\equiv S_H^*$ (Silverman[10]); H_λ^k(Darus and Al Shaqsi [7]) and $H(\lambda)$(Yalçin and Öztürk][11]).

(ii) $H_0^1(1,\alpha)\equiv N_H(\alpha)$ (Ahuja and Jahangiri [1]);

(iii) $H_0^1(p,\alpha)\equiv H_\lambda^k(p,\alpha)$ (Al Shaqsi and Darus [2]);

(iv) $H_0^{\beta}(p,\alpha)\equiv H(p,\alpha,\beta)$(Atshan et al. [4]).

Also, we note that the analytic part of the class $H_0^1(p,\alpha)$ was introduced and studied by Goel and Sohi [8].

Also, several authors studied some same properties of various other classes, like, Aouf et al. [3], Joshi and Sangle [9] and Atshan and Wanas [5].

We further denote by $A_q^{\beta}(p,\alpha)$ the subclass of $H_q^{\beta}(p,\alpha)$ that satisfies the relation

$$A_q^{\beta}(p,\alpha) = A(p)\cap H_q^{\beta}(p,\alpha). \qquad (1.4)$$

Coefficient bounds

First, we determine the sufficient condition for $f=h+\bar{g}$ to be in the class $H_q^{\beta}(p,\alpha)$.

Theorem 1. Let $f=h+\bar{g}$ (h and g being given by (1.1)). If

$$\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\,|a_n| + \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\,|b_n|$$
$$\leq \delta(p,q+1)-\alpha, \qquad (2.1)$$

where $\beta \geq 0,\ 0 \leq \alpha < p,\ p \in N,\ q \in N_0$ and $p > q$, then f is harmonic p-valent sense-preserving in U and $f \in H_q^{\beta}(p,\alpha)$.

Proof. Let $w(z) = (1-\beta)\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} + \beta\frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-(q+1)}}$. To prove that $Re\{w(z)\} \geq \frac{\alpha}{\delta(p,q+1)}$, it is sufficient to show equivalently that

$$|(p-\alpha)+\delta(p,q+1)w(z)| \geq |(p+\alpha)-\delta(p,q+1)w(z)|.$$

Substituting for $w(z)$ and resorting to simple calculations, we find that

$$|(p-\alpha)+\delta(p,q+1)w(z)|$$

$$= \left|(p-\alpha) + \delta(p,q+1)\left[(1-\beta)\frac{\left(\delta(p,q)z^{p-q} + \sum_{n=p+1}^{\infty}\delta(n,q)\,a_n\,z^{n-q} + \sum_{n=p}^{\infty}\delta(n,q)\,b_n\,(\overline{z})^{n-q}\right)}{\delta(p,q)z^{p-q}} + \beta\frac{\left(\delta(p,q+1)z^{p-(q+1)} + \sum_{n=p+1}^{\infty}\delta(n,q+1)a_n z^{n-(q+1)} + \sum_{n=p}^{\infty}\delta(n,q+1)b_n(\overline{z})^{n-(q+1)}\right)}{\delta(p,q+1)z^{p-(q+1)}}\right]\right|$$

$$\geq (p+\delta(p,q+1)-\alpha) - \sum_{n=p+1}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,|a_n||z|^{n-p} - \sum_{n=p}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,|b_n|\,|z|^{n-p} \qquad (2.2)$$

and

$$|(p+\alpha)-\delta(p,q+1)w(z)|$$

$$= \Bigg| (p+\alpha) - \delta(p,q+1)\left[(1-\beta)\frac{\left(\delta(p,q)z^{p-q} + \sum_{n=p+1}^{\infty} \delta(n,q)\, a_n\, z^{n-q} + \sum_{n=p}^{\infty} \delta(n,q)\, b_n\, (\overline{z})^{n-q}\right)}{\delta(p,q)z^{p-q}} + \beta\frac{\left(\delta(p,q+1)z^{p-(q+1)} + \sum_{n=p+1}^{\infty} \delta(n,q+1)a_n z^{n-(q+1)} + \sum_{n=p}^{\infty} \delta(n,q+1)b_n(\overline{z})^{n-(q+1)}\right)}{\delta(p,q+1)z^{p-(q+1)}}\right]\Bigg|$$

$$\leq (p-\delta(p,q+1)+\alpha) + \sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\,|a_n|\,|z|^{n-p} + \sum_{n=p}^{\infty} \delta(n,q)[(p-q) + \beta(n-p)]\,|b_n||z|^{n-p}\,, \qquad (2.3)$$

from (2.2) and (2.3), we have

$$|(p-\alpha)+\delta(p,q+1)w(z)| - |(p+\alpha)-\delta(p,q+1)w(z)|$$

$$> 2\left\{(\delta(p,q+1)-\alpha) - \sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\,|a_n| - \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\,|b_n|\right\} \geq 0.$$

The harmonic functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{x_n}{\delta(n,q)[(p-q)+\beta(n-p)]}\, z^n + \sum_{n=p}^{\infty} \frac{\overline{y_n}}{\delta(n,q)[(p-q)+\beta(n-p)]}\, (\overline{z})^n \qquad (2.4)$$

$\left(\sum_{n=p+1}^{\infty}|x_n|+\sum_{n=p}^{\infty}|y_n|=\delta(p,q+1)-\alpha\right)$ show that the coefficient bound given by (2.1)is sharp.

The functions of the form (2.4) are in $H_q^{\beta}(p,\alpha)$ because in view of (2.1), we have

$$\sum_{n=p+1}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,|a_n|+\sum_{n=p}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,|b_n|$$

$$=\sum_{n=p+1}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)].\ \frac{|x_n|}{\delta(n,q)[(p-q)+\beta(n-p)]}$$

$$+\sum_{n=p}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)].\ \frac{|y_n|}{\delta(n,q)[(p-q)+\beta(n-p)]}$$

$$=\sum_{n=p+1}^{\infty}|x_n|+\sum_{n=p}^{\infty}|y_n|=\delta(p,q+1)-\alpha.$$

Now, we need to prove that the condition (2.1) is also necessary for functions of (1.2) to be in the class $A_q^{\beta}(p,\alpha)$.

Theorem 2. Let $=h+\bar{g}$, where h and g are given by (1.2). Then $f\in A_q^{\beta}(p,\alpha)$ if and only if

$$\sum_{n=p+1}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,a_n+\sum_{n=p}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,b_n$$

$$\leq\delta(p,q+1)-\alpha, \qquad (2.5)$$

where $\beta\geq 0, 0\leq\alpha<p, p\in N, q\in N_0$ and $p>q$

Proof. Since $A_q^{\beta}(p,\alpha)\subset H_q^{\beta}(p,\alpha)$, we only need to prove the "only if" part of this theorem.

Let $f(z)\in A_q^{\beta}(p,\alpha)$. Using (1.3), we get

$$Re\left\{\delta(p,q+1)\left[(1-\beta)\left(\frac{h^{(q)}(z)+\overline{g^{(q)}(z)}}{\delta(p,q)z^{p-q}}\right)+\beta\left(\frac{h^{(q+1)}(z)+\overline{g^{(q+1)}(z)}}{\delta(p,q+1)z^{p-(q+1)}}\right)\right]\right\}$$

$$= Re\left\{\delta(p,q+1)(1-\beta)\left(\frac{\left(\delta(p,q)z^{p-q}-\sum_{n=p+1}^{\infty}\delta(n,q)a_n z^{n-q}-\sum_{n=p}^{\infty}\delta(n,q)b_n(\overline{z})^{n-q}\right)}{\delta(p,q)z^{p-q}}\right)\right.$$

$$\left.+\delta(p,q+1)\beta\left(\frac{\left(\delta(p,q+1)z^{p-(q+1)}-\sum_{n=p+1}^{\infty}\delta(n,q+1)a_n z^{n-(q+1)}-\sum_{n=p}^{\infty}\delta(n,q+1)b_n(\overline{z})^{n-(q+1)}\right)}{\delta(p,q+1)z^{p-(q+1)}}\right)\right\}$$

$$= Re\left\{\delta(p,q+1)-\sum_{n=p+1}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,a_n\,z^{n-p}\right.$$
$$\left.-\sum_{n=p}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,b_n\,(\overline{z})^{n-p}\right\}\geq\alpha.$$

If we choose z to be real and $z\to 1^-$, we get

$$\delta(p,q+1)-\sum_{n=p+1}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,a_n$$
$$-\sum_{n=p}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,b_n\geq\alpha.$$

Hence

$$\sum_{n=p+1}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,a_n+\sum_{n=p}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,b_n$$
$$\leq\delta(p,q+1)-\alpha,$$

which completes the proof of Theorem 2.

Some properties for the class $A_q^{\beta}(p,\alpha)$

Extreme points for the class $A_q^{\beta}(p,\alpha)$ are given in the following Theorem.

Theorem 3. Let the function $f(z)$ be given by (1.2). Then $f(z)\in A_q^{\beta}(p,\alpha)$ if and only if

$$f(z)=\sum_{n=p}^{\infty}\left(\lambda_n h_n(z)+\mathcal{E}_n g_n(z)\right), \qquad (3.1)$$

where $z\in U,\ h_p(z)=z^p$,

$$h_n(z) = z^p - \frac{\delta(p,q+1)-\alpha}{\delta(n,q)[(p-q)+\beta(n-p)]} z^n, \qquad (n = p+1, p+2, \dots) \qquad (3.2)$$

and

$$g_n(z) = z^p - \frac{\delta(p,q+1)-\alpha}{\delta(n,q)[(p-q)+\beta(n-p)]} (\overline{z})^n, \qquad (n = p, p+1, \dots), \qquad (3.3)$$

$$\sum_{n=p}^{\infty} (\lambda_n + \mathcal{E}_n) = 1 \ (\lambda_n \geq 0, \mathcal{E}_n \geq 0).$$

In particular, the extreme points of $A_q^{\beta}(p,\alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Suppose $f(z)$ is of the form (3.1). Using (3.2)and (3.3), we get

$$f(z) = \sum_{n=p}^{\infty} \left(\lambda_n h_n(z) + \mathcal{E}_n g_n(z)\right)$$

$$= \sum_{n=p}^{\infty} (\lambda_n + \mathcal{E}_n) z^p - \sum_{n=p+1}^{\infty} \frac{\delta(p,q+1)-\alpha}{\delta(n,q)[(p-q)+\beta(n-p)]} \lambda_n z^n$$
$$- \sum_{n=p}^{\infty} \frac{\delta(p,q+1)-\alpha}{\delta(n,q)[(p-q)+\beta(n-p)]} \mathcal{E}_n (\overline{z})^n$$
$$= z^p - \sum_{n=p}^{\infty} \frac{\delta(p,q+1)-\alpha}{\delta(n,q)[(p-q)+\beta(n-p)]} \lambda_n (z)^n$$
$$- \sum_{n=p}^{\infty} \frac{\delta(p,q+1)-\alpha}{\delta(n,q)[(p-q)+\beta(n-p)]} \mathcal{E}_n (\overline{z})^n,$$

then

$$\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)] \frac{\delta(p,q+1)-\alpha}{\delta(n,q)[(p-q)+\beta(n-p)]} \lambda_n$$

$$+ \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)] \frac{\delta(p,q+1)-\alpha}{\delta(n,q)[(p-q)+\beta(n-p)]} \mathcal{E}_n$$

$$= (\delta(p,q+1)-\alpha)\left[\sum_{n=p}^{\infty}(\lambda_n+\mathcal{E}_n)-\lambda_p\right] = (\delta(p,q+1)-\alpha)[1-\lambda_p]$$

$$\leq \delta(p,q+1)-\alpha$$

and so $f(z)\in A_q^{\beta}(p,\alpha)$.

Conversely, assume that $f(z)\in A_q^{\beta}(p,\alpha)$. Then

$$a_n \leq \frac{\delta(p,q+1)-\alpha}{\delta(n,q)[(p-q)+\beta(n-p)]}\lambda_n,$$

and

$$b_n \leq \frac{\delta(p,q+1)-\alpha}{\delta(n,q)[(p-q)+\beta(n-p)]}\mathcal{E}_n.$$

Set

$$\lambda_n = \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}a_n \quad (n=p+1,p+2,\dots),$$

$$\mathcal{E}_n = \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}b_n \quad (n=p,p+1,\dots),$$

where $\sum_{n=p}^{\infty}(\lambda_n+\mathcal{E}_n)=1$, we have

$$f(z)=z^p-\sum_{n=p+1}^{\infty}a_n z^n-\sum_{n=p}^{\infty}b_n(\overline{z})^n$$

$$=z^p-\sum_{n=p+1}^{\infty}\frac{\delta(p,q+1)-\alpha}{\delta(n,q)[(p-q)+\beta(n-p)]}\lambda_n z^n$$

$$-\sum_{n=p}^{\infty}\frac{\delta(p,q+1)-\alpha}{\delta(n,q)[(p-q)+\beta(n-p)]}\mathcal{E}_n(\overline{z})^n.$$

$$=z^p-\sum_{n=p+1}^{\infty}(z^p-h_n(z))\lambda_n-\sum_{n=p}^{\infty}(z^p-g_n(z))\mathcal{E}_n$$

$$=\left[1-\left(\sum_{n=p+1}^{\infty}\lambda_n+\sum_{n=p}^{\infty}\mathcal{E}_n\right)\right]z^p+\sum_{n=p+1}^{\infty}\lambda_n h_n(z)+\sum_{n=p}^{\infty}\mathcal{E}_n g_n(z)$$

$$= \lambda_p\, h_p(z) + \sum_{n=p+1}^{\infty} \lambda_n\, h_n(z) + \sum_{n=p}^{\infty} \mathcal{E}_n\, g_n(z)$$

$$= \sum_{n=p}^{\infty} (\lambda_n\, h_n(z) + \mathcal{E}_n\, g_n(z)).$$

This completes the proof of Theorem 3.

The next, we get the distortion Theorem.

Theorem 4. Let the function $f(z)$ defined by (1.2) be in the class $A_q^{\beta}(p,\alpha)$. Then for $|z| = r < 1$, we have

$$|f(z)| \le (1+b_p)r^p + \left[\frac{\delta(p,q+1)-\alpha}{\delta(p+1,q)[p-q+\beta]} - \frac{\delta(p,q)(p-q)b_p}{\delta(p+1,q)[p-q+\beta]}\right] r^{p+1}, \tag{3.4}$$

and

$$|f(z)| \ge (1-b_p)r^p - \left[\frac{\delta(p,q+1)-\alpha}{\delta(p+1,q)[p-q+\beta]} - \frac{\delta(p,q)(p-q)b_p}{\delta(p+1,q)[p-q+\beta]}\right] r^{p+1}. \tag{3.5}$$

The equalities in (3.4)and (3.5) are attained for the functions $f(z)$ given by

$$f(z) = z^p + b_p(\overline{z})^p + \left[\frac{\delta(p,q+1)-\alpha}{\delta(p+1,q)[p-q+\beta]} - \frac{\delta(p,q)(p-q)b_p}{\delta(p+1,q)[p-q+\beta]}\right] (\overline{z})^{p+1},$$

and

$$f(z) = z^p - b_p(\overline{z})^p - \left[\frac{\delta(p,q+1)-\alpha}{\delta(p+1,q)[p-q+\beta]} - \frac{\delta(p,q)(p-q)b_p}{\delta(p+1,q)[p-q+\beta]}\right] (\overline{z})^{p+1}.$$

Proof. Suppose $(z) \in A_q^{\beta}(p,\alpha)$. Using (1.1)and (2.1) of Theorem 1, we find that

$$|f(z)| \le (1+b_p)r^p + \sum_{n=p+1}^{\infty} (a_n + b_n)\, r^n \le (1+b_p)r^p + \sum_{n=p+1}^{\infty} (a_n + b_n)\, r^{p+1}$$

$$= (1+b_p)r^p + \frac{\delta(p,q+1)-\alpha}{\delta(p+1,q)[p-q+\beta]} \times \sum_{n=p+1}^{\infty} \frac{\delta(p+1,q)[p-q+\beta]}{\delta(p,q+1)-\alpha} (a_n + b_n)\, r^{p+1}$$

$$\leq (1+b_p)r^p + \frac{\delta(p,q+1)-\alpha}{\delta(p+1,q)[(p-q)+\beta]} \times$$

$$\sum_{n=p+1}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha} (a_n + b_n)\, r^{p+1}$$

$$\leq (1+b_p)r^p + \frac{\delta(p,q+1)-\alpha}{\delta(p+1,q)[p-q+\beta]}\left[1 - \frac{\delta(p,q)(p-q)b_p}{\delta(p,q+1)-\alpha}\right] r^{p+1}$$

$$= (1+b_p)r^p + \left[\frac{\delta(p,q+1)-\alpha}{\delta(p+1,q)[p-q+\beta]} - \frac{\delta(p,q)(p-q)b_p}{\delta(p+1,q)[p-q+\beta]}\right] r^{p+1}.$$

Similarly, we can prove the left- hand inequality.
This completes the proof of Theorem 4.

Theorem 5. The class $A_q^{\beta}(p,\alpha)$ is convex set.

Proof. Let the function $f_j(z)$ $(j=1,2)$ be in the class $A_q^{\beta}(p,\alpha)$. It is sufficient to show that the function h defined by

$$h(z) = (1-t)\, f_1(z) + t\, f_2(z), (0 \leq t < 1)$$

is in the class $A_q^{\beta}(p,\alpha)$,
where

$$f_j(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,j}\, z^n - \sum_{n=p}^{\infty} b_{n,j}\, (\overline{z})^n, \ \ (j=1,2)$$

since for $0 \leq t \leq 1$,

$$h(z) = z^p - \sum_{n=p+1}^{\infty} \left((1-t)a_{n,1} - t a_{n,2}\right) z^n - \sum_{n=p}^{\infty} \left((1-t)b_{n,1} - t b_{n,2}\right) (\overline{z})^n.$$

In view of Theorem 2, we have

$$\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\left((1-t)a_{n,1} - t a_{n,2}\right)$$

$$+ \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\left((1-t)b_{n,1} - t b_{n,2}\right)$$

$$= (1-t)\left(\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]a_{n,1} + \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]b_{n,1}\right)$$

$$+t\left(\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]a_{n,2} + \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]b_{n,2}\right)$$

$$\leq (1-t)[\delta(p,q+1)-\alpha] + t\,[\delta(p,q+1)-\alpha] = \delta(p,q+1)-\alpha.$$

Hence $h(z) \in A_q^{\beta}(p,\alpha)$.

We define the convolution two harmonic functions $f(z)$ and $F(z)$ by

$$(f * F)(z) = z^p - \sum_{n=p+1}^{\infty} a_n A_n\, z^n - \sum_{n=p}^{\infty} b_n B_n\, (\overline{z})^n, \qquad (3.6)$$

where

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n\, z^n - \sum_{n=p}^{\infty} b_n\, (\overline{z})^n,$$

and

$$F(z) = z^p - \sum_{n=p+1}^{\infty} A_n\, z^n - \sum_{n=p}^{\infty} B_n\, (\overline{z})^n.$$

Theorem 6. For $0 \leq \sigma \leq \alpha < p$, let $f(z) \in A_q^{\beta}(p,\alpha)$ and $F(z) \in A_q^{\beta}(p,\sigma)$. Then

$(f * F)(z) \in A_q^{\beta}(p,\alpha) \subset A_q^{\beta}(p,\sigma)$.

Proof. Let the convolution $(f * F)(z)$ be of the form (3.6), Then we want to prove that the coefficient of $(f * F)(z)$ satisfy the condition of the Theorem 2.

Since $f(z) \in A_q^{\beta}(p,\alpha)$ and $(z) \in A_q^{\beta}(p,\sigma)$, Then by Theorem 2, we have

$$\sum_{n=p+1}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha} a_n + \sum_{n=p}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha} b_n \leq 1, \quad (3.7)$$

and

$$\sum_{n=p+1}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\sigma}A_n+\sum_{n=p}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\sigma}B_n \leq 1. \quad (3.8)$$

From (3.8), we obtain the following inequalities

$$A_n<\frac{\delta(p,q+1)-\sigma}{\delta(n,q)[(p-q)+\beta(n-p)]} \quad (n=p+1,p+2,\ldots),$$

$$B_n<\frac{\delta(p,q+1)-\sigma}{\delta(n,q)[(p-q)+\beta(n-p)]} \quad (n=p,p+1,p+2,\ldots).$$

Therefore,

$$\sum_{n=p+1}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}a_nA_n+\sum_{n=p}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}b_n\,B_n$$

$$=\frac{\delta(p,q)(p-q)}{\delta(p,q+1)-\alpha}b_p\,B_p+\sum_{n=p+1}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}a_nA_n+\sum_{n=p+1}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}b_n\,B_n$$

$$<\frac{[\delta(p,q)(p-q)][\delta(p,q+1)-\sigma]}{[\delta(p,q+1)-\alpha][\delta(p,q)(p-q)]}b_p+\sum_{n=p+1}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)][\delta(p,q+1)-\sigma]}{[\delta(p,q+1)-\alpha](\delta(n,q)[(p-q)+\beta(n-p)])}a_n+\sum_{n=p+1}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)][\delta(p,q+1)-\sigma]}{[\delta(p,q+1)-\alpha](\delta(n,q)[(p-q)+\beta(n-p)])}b_n$$

$$=\sum_{n=p+1}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)][\delta(p,q+1)-\sigma]}{[\delta(p,q+1)-\alpha](\delta(n,q)[(p-q)+\beta(n-p)])}a_n$$

$$+\sum_{n=p}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)][\delta(p,q+1)-\sigma]}{[\delta(p,q+1)-\alpha](\delta(n,q)[(p-q)+\beta(n-p)])}b_n$$

$$\leq \sum_{n=p+1}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}a_n+\sum_{n=p}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}b_n \leq 1.$$

Then

$(f*F)(z)\in A_q^{\beta}(p,\alpha)\subset A_q^{\beta}(p,\sigma)$, and the proof is complete.

Next, theorem to prove the class $A_q^{\beta}(p,\alpha)$ is closed under convex combination.

Theorem 7. Let $0\leq c_i<1$ for $i=1,2,\dots$ and $\sum_{i=1}^{\infty}c_i=1$, if the functions $f(z)$ defined by

$$f_i(z)=z^p-\sum_{n=p+1}^{\infty}a_{n,i}\,z^n-\sum_{n=p}^{\infty}b_{n,i}\,(\overline{z})^n,\ \ (z\in U, i=1,2,\dots),$$

are in the class $A_q^{\beta}(p,\alpha)$, for every $i=1,2,\dots$, then $\sum_{i=1}^{\infty}c_i\,f_i(z)$ of the form

$$\sum_{i=1}^{\infty}c_i\,f_i(z)=z^p-\sum_{n=p+1}^{\infty}\left(\sum_{i=1}^{\infty}c_i\,a_{n,i}\right)z^n-\sum_{n=p}^{\infty}\left(\sum_{i=1}^{\infty}c_i\,b_{n,i}\right)(\overline{z})^n,$$

is in the class $A_q^{\beta}(p,\alpha)$.

Proof. Since $f_i(z)\in A_q^{\beta}(p,\alpha)$, it follows from Theorem 2 that

$$\sum_{n=p+1}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]a_{n,i}$$

$$+\sum_{n=p}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]b_{n,i}\leq\delta(p,q+1)-\alpha,$$

for every $i=1,2,\dots$. Hence

$$\sum_{n=p+1}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\left(\sum_{i=1}^{\infty}c_i\,a_{n,i}\right)$$

$$+\sum_{n=p}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\left(\sum_{i=1}^{\infty}c_i\,b_{n,i}\right)$$

$$= \sum_{i=1}^{\infty} c_i \left(\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]a_{n,i} + \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]b_{n,i} \right)$$

$$\leq \sum_{i=1}^{\infty} c_i \; (\delta(p,q+1)-\alpha) = \delta(p,q+1)-\alpha.$$

Thus $\sum_{i=1}^{\infty} c_i f(z) \in A_q^{\beta}(p,\alpha)$ and the proof is complete.

Reference

[1] O. Ahuja and J. Jahangiri, Noshiro-type harmonic univalent functions, Sci. Math. Japan, 65 (2002), 293–299.
[2] K. Al Shaqsi and M. Darus, A new class of multivalent harmonic functions, General Math., 14 (2006), 37–46.
[3] M. K. Aouf, A. A. Shamandy, A. O. Mostafa and A. K. Wagdy, A new class of harmonic
p-valent functions of complex order, proceedings of Pakistan Academy of Sciences, 49 (3)(2012), 219-225.
[4] W. G. Atshan, S. R. Kulkaini and R. K. Raina, A class of multivalent harmonic functions involving a generalized Ruscheweyh type operator, Matematicki vesink, 60(2008), 207-213.
[5] W. G. Atshan and A. K. Wanas, On a new class of harmonic univalent functions, Matematicki vesink, 65(2013), 1-10.
[6] J. Clunie and T. Shell-Small, Harmonic univalent functions, Ann. Acad. Aci. Fenn. Ser. AI Math., 9 (1984), 3–25.
[7] M. Darus and K. Al Shaqsi, On harmonic univalent functions defined by a generaliz-zed Ruscheweyh derivatives operator, Lobachevskii J. Math., 22 (2006), 19–26.
[8] R. Goel and N. Sohi, New criteria for p-valent, Indian J. Pure Appl. Math., 7(2004), 55–61.
[9] S. B. Joshi and N. D. Sangle, New subclass of Goodman-type p-valent harmonic fun ctions, Facully of Sciences and Math., Unversity of Niš, Serbia, 22(1)(2008), 193-204.
[10] H. Silverman, Harmonic univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51 (1998), 283–289.
[11] S. Yalçin and M. Öztürk, A new subclass of complex harmonic functions, Math. Inequal. Appl., 7 (2004), 55–61.